Complexity of frustration

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Q-team organization chart

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Overview of the talk

1. Topological frustration in spin chains
2. Excess of entanglement
3. Entanglement cooling and robustness
4. Complexity of entanglement spectrum
5. Topological frustration is magical!


arXiv > quant-ph > arXiv:2209.10541

Quantum Physics

[Submitted on 21 Sep 2022]

Complexity of frustration: a new source of non-local non-stabilizerness

J. Odavić, T. Haug, G. Torre, A. Hamma, F. Franchini, S. M. Giampaolo


arXiv > quant-ph > arXiv:2210.13495

Quantum Physics

[Submitted on 24 Oct 2022]

Random unitaries, Robustness, and Complexity of Entanglement

J. Odavić, G. Torre, N. Mijić, D. Davidović, F. Franchini, S. M. Giampaolo

Topologically frustrated models in 1d

3 year long frustration

1. Fundamental ideas
- Exact correlation functions
- Site-dependent (incommensurate) order parameter
- Boundary-dependent phase diagram

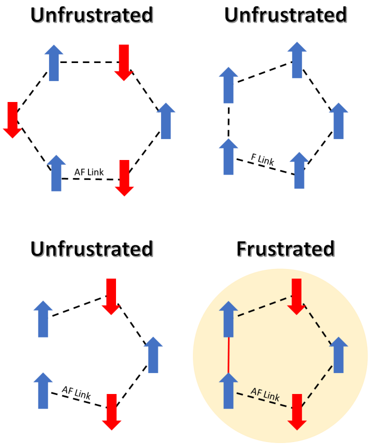
2. Phenomenology
- Excess of entanglement
- Non-stabilizerness and magic
- Topological properties
- Non-trivial dynamical response (Loschmidt echo)

3. Applications
- Quantum batteries
- Applications in quantum computing

- arXiv.2210.13495 (2022)
- arXiv:2209.10541 (2022)
- Phys. Rev. B **106**, 125145 (2022)
- Phys. Rev. B **105**, 184424 (2022)
- SciPost Physics **12**, 075 (2022)
- Phys. Rev. B **105**, 064408 (2022)
- Sci. Rep. **11**, 6508 (2021)
- Phys. Rev. B **103**, 014429 (2021)
- Comm. Phys. **3**, 220 (2020)
- NJP **22**, 083024 (2020)
- J. Phys. Comm. **3**, 081001 (2019)

1a. Topological frustration via FBC

...in classical Ising spins



Our focus

- Non-extensive (single-loop) geometrical frustration
- Spatially invariant one-dimensional systems with
  1. Periodic boundary conditions
  2. Odd number of spins
  3. Antiferromagnetic coupling

  we denote as frustrated boundary conditions (FBC)

1b. Entanglement properties and different diagnostics

We consider 1d transverse field Ising model (TFIM)

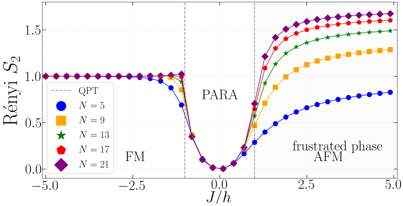
$$H = J \sum_{i=1}^{L} \sigma_i^x \sigma_{i+1}^x - \lambda \sum_{i=1}^{L} \sigma_i^z .$$

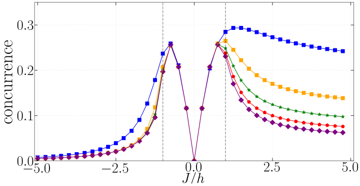
- **Rényi-α entanglement entropy**

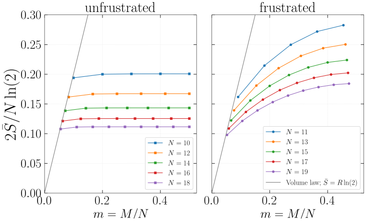
$$S_\alpha(\rho_A) = \frac{1}{1-\alpha} \log_2 \mathrm{Tr}\,[\rho_A^\alpha], \quad \text{with} \quad \alpha \in [0,1) \cup (1,\infty]$$

- **Von Neumann entanglement entropy** (Rényi $\alpha \to 1$)
- **Nearest-neighbor concurrence** (short-range entanglement)

1b. Entanglement properties







→ experimentally observed[1]

[1] Monroe et al.; Nature **465**, 590–593 (2010)

1c. Explanation

Ground state manifolds at the Ising classical point $h = 0$

$$H = J \sum_{i=1}^{N} \sigma_i^x \sigma_{i+1}^x, \qquad L = 2M + 1, M \in \mathbb{N}$$

Unfrustrated ferromagnetic ($J = -1$)

Finite degeneracy $\rightarrow 2$

$$|- - ... -\rangle$$
$$|+ + ... +\rangle$$

corresponding to N spins where

$$|\pm\rangle = \frac{1}{\sqrt{2}} (|0\rangle \pm |1\rangle)$$

Frustrated antiferromagnetic ($J = 1$)

Extensive degeneracy $\rightarrow 2N$

$$\left.\begin{array}{l} |++-+-...\rangle \\ |-++-+...\rangle \\ |+-++-...\rangle \\ \vdots \end{array}\right\} = N \text{ times}$$

$$\left.\begin{array}{l} |--+-+...\rangle \\ |+--+-...\rangle \\ |-+--+...\rangle \\ \vdots \end{array}\right\} = N \text{ times}$$

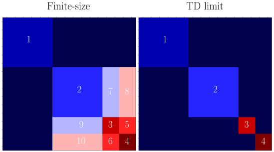
Domain wall embedded into Néel states!

1c. Explanation - Reduced density matrix

1. The ground state at $h \to 0^+$ can be represented as a linear superposition of kink states

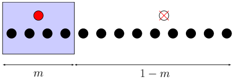
$$|W_k\rangle = \frac{1}{\sqrt{N}} \left(|{++} - + - \ldots\rangle + |- {++} - + \ldots\rangle + |+ - {++} - \ldots\rangle + \ldots \right),$$

with exact half-chain reduced density matrix $\rho^{\text{frus}}_{[m=N/2]}$



* 1 and 2: Néel orders, 3: kink even site, 4: kink odd site, dark blue $\to$ zeroes

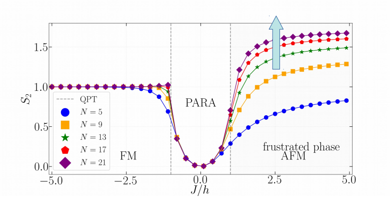
2. Semi-classical picture of a quasiparticle[2]



$$\rho^{\text{frus}}_{[m]} = \rho^{\text{unfrus}}_{[m]} \otimes \rho^{\text{semi}}_{[m]}, \qquad \text{where} \qquad \rho^{\text{semi}}_{[m]} = m|0\rangle\langle 0| + (1-m)|1\rangle\langle 1|.$$

[2] Giampaolo, Ramos, and Franchini; J. Phys. Comm. **3**, 081001 (2019)

1d. Exact results in the thermodynamic limit

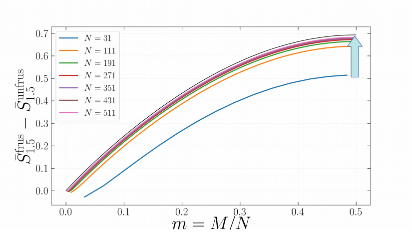


We obtain for the Rényi-α entanglement[3,4]

$$S_\alpha(\rho^{\text{frus}}_{[m]}) = \frac{1}{1-\alpha}\log\left(m^\alpha + (1-m)^\alpha\right) + \log 2,$$

and in the limit $\alpha \to 1$ the von Neumann

$$S_1(\rho^{\text{frus}}_{[m]}) = -m\log(m) - (1-m)\log(1-m) + \log 2.$$



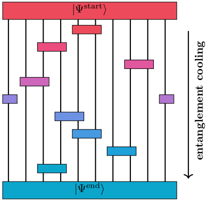
- quasiparticle in the ground state!
- excess of long-range entanglement (beyond area-law)!
- entanglement immune to the introducing integrability breaking terms!

[3] Castro-Alvaredo, De Fazio, Doyon, and Szécsényi; Phys. Rev. Lett. **121**, 170602 (2018); JHEP **39** (2018);JHEP **58** (2019).

[4] You, Wybo, Pollmann, and Sondhi; Phys. Rev. B **106**, L161104 (2022).

2a. How robust (stochastically irreversible) is this?

- We attempt to disentangle the frustrated ground state using the entanglement cooling



- Simulated annealing Metropolis Monte-Carlo quantum circuit[5]

Set 1	Set 2
$h_j^{(1)} = \sigma_j^z \otimes \mathbb{I}_{j+1} + \mathbb{I}_j \otimes \sigma_{j+1}^z$	$h_j^{(4)} = \sigma_j^x \otimes \mathbb{I}_{j+1} + \mathbb{I}_j \otimes \sigma_{j+1}^x$
$h_j^{(2)} = \sigma_j^x \otimes \sigma_{j+1}^x$	$h_j^{(5)} = \sigma_j^y \otimes \mathbb{I}_{j+1} + \mathbb{I}_j \otimes \sigma_{j+1}^y$
$h_j^{(3)} = \sigma_j^y \otimes \sigma_{j+1}^y$	$h_j^{(6)} = \sigma_j^z \otimes \sigma_{j+1}^z$

- Set 1 is parity preserving
- Sets 1 & 2 are taken together from the universal set[6]

- focus on Rényi-2 due to less computational demand
- use GPU parallel code[7]

5 Yang, Hamma, Giampaolo, Mucciolo, and Chamon, Phys. Rev. B **96**, 020408 (2017)

6 Barenco, Bennett, Cleve, DiVincenzo, Margolus, Shor, Sleator, Smolin, and Weinfurter, Physical Review A **52**, 3457 (1995).

7 N. Mijić, and D. Davidović; arXiv:2203.09353 (2022).

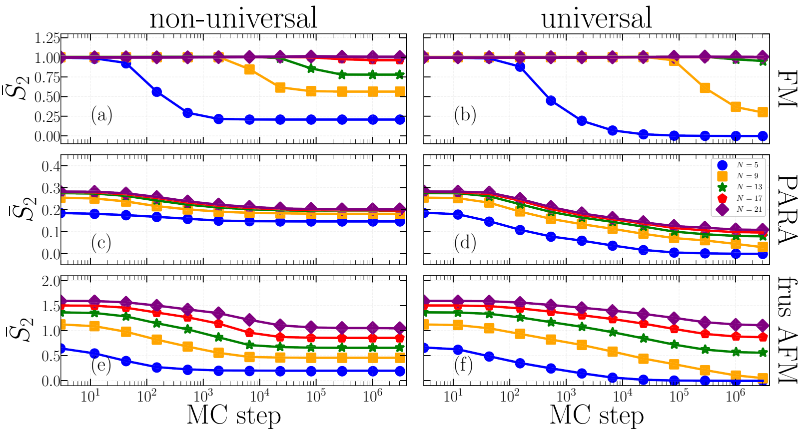
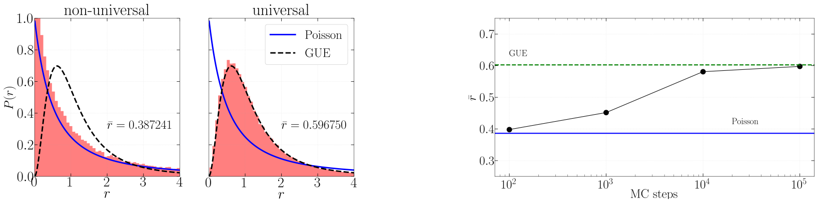


Figure: Averaged half-chain Rényi-2 entanglement entropy during the entanglement cooling over $M = 96$ Metropolis MC trajectories for ground states of the TFIM Hamiltonian different macroscopic phases.

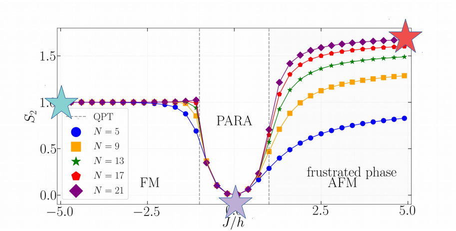
2c. Entanglement spectrum complexity - arXiv.2210.13495



- Consecutive entanglement spectrum spacing ratio histogram and average at the end of the cooling algorithm.
- Frustrated ground state starting point.

Set 1	Set 2
$h_j^{(1)} = \sigma_j^z \otimes \mathbb{I}_{j+1} + \mathbb{I}_j \otimes \sigma_{j+1}^z$ $h_j^{(2)} = \sigma_j^x \otimes \sigma_{j+1}^x$ $h_j^{(3)} = \sigma_j^y \otimes \sigma_{j+1}^y$	$h_j^{(4)} = \sigma_j^x \otimes \mathbb{I}_{j+1} + \mathbb{I}_j \otimes \sigma_{j+1}^x$ $h_j^{(5)} = \sigma_j^y \otimes \mathbb{I}_{j+1} + \mathbb{I}_j \otimes \sigma_{j+1}^y$ $h_j^{(6)} = \sigma_j^z \otimes \sigma_{j+1}^z$

3a. Quantum information perspective



Limits

- FM Greenberger–Horne–Zeilinger state

$$|GHZ\rangle = \frac{1}{\sqrt{2N}} \left(|+\rangle^{\otimes N} + |-\rangle^{\otimes N} \right)$$

- PARA

$$|\psi\rangle = | + \ \text{ or } \ -\rangle^{\otimes N}$$

- frustrated AFM W-state

$$|W\rangle = \frac{1}{\sqrt{N}} \left(|100...0\rangle + |010...0\rangle + ... + |000...1\rangle \right)$$

Wait, but HOW?!

3b. Transforming $|W_k\rangle$ in a $|W\rangle$ state - arXiv:2209.10541

$$|W_k\rangle = \hat{\mathcal{S}}\,|W\rangle$$

$$|W\rangle = \frac{1}{\sqrt{N}}\left(|100...0\rangle + |010...0\rangle + ... + |000...1\rangle\right)$$

$$|W_k\rangle = \frac{1}{\sqrt{N}}\left(|{+}{+}-+-...\rangle + |-{+}{+}-+...\rangle + |+-{+}{+}-...\rangle + ...\right),$$

$|W\rangle$ retain the maximum amount of b. entanglement after local measurement on one of its part.

$$\hat{\mathcal{S}} = \prod_{i=1}^{N-1} \mathsf{C}(N, N-i)\left(\prod_{i=1}^{M} \sigma^z_{2i-1}\right)\mathsf{H}(N)\sigma^z_N \prod_{i=1}^{N-1} \mathsf{C}(i, i+1)\Pi^z$$

Clifford gates (Clifford circuits)

- $\mathsf{H}(i)$ – Hadamard Gate
- $\mathsf{C}(i, j)$ – C-Not Gate
- $\Pi^z = \bigotimes_{i=1}^{N} \sigma^z_i$

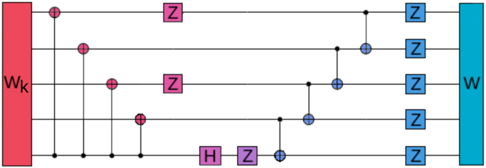


Figure: Clifford circuit $\hat{\mathcal{S}}$ for $N = 5$.

- **These two states are of equal complexity!**

4a. A measure of complexity: non-stabilizerness (*magic*)

- Quantify the "distance" between $|\psi\rangle$ and the set of states that can be obtained with a Clifford Circuit starting from a fully separable state

Stabilizer Rényi-2 Entropy (SRE)[8]

$$\mathcal{M}_2(|\psi\rangle) = -\log_2\left(\frac{1}{2^N}\sum_P \langle\psi| P |\psi\rangle^4\right), \quad \text{Pauli strings} \quad P = \bigotimes_{j=1}^{N} P_j; \quad P_j \in \{\sigma_j^0, \sigma_j^x, \sigma_j^y, \sigma_j^z\}$$

- *Clifford Circuits* can be efficiently simulated in a classical computer (no quantum advantage) $\rightarrow$ Gottesman-Knill theorem[9].

- Example $N = 2$ (**16** Paulies), $N = 3$ (**64** Paulies), $N = 4$ (**256** Paulies), ...
- 4^N possible operators
- non-variational probe of complexity

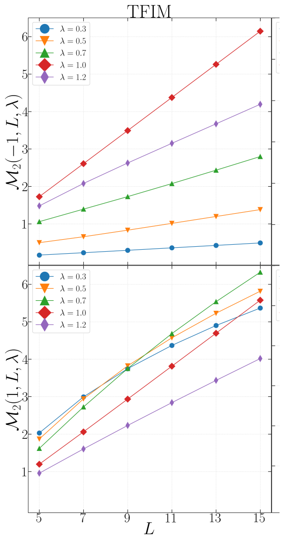
8 Leone, Oliviero, and Hamma; Phys. Rev. Lett. **128**, 050402 (2022)

9 Aaronson, and Gottesman; Phys. Rev. A **70**, 052328 (2004)

4b. Numerical SRE for TFIM - arXiv:2209.10541

$$H = J \sum_{i=1}^{N} \sigma_i^x \sigma_{i+1}^x - \lambda \sum_{i=1}^{N} \sigma_i^z .$$

- Linear scaling of SRE with spin/qubit number N in unfrustrated regime[10]
- Slope maximal at QPT
- SRE vanishes going to classical point $h \to 0$
- Frustrated state linear + extra ?
- **We obtain analytic results for extra!**



[10] Oliviero, Leone, and Hamma; Phys. Rev. A **106**, 042426 (2022)

4c. SRE of a $|W\rangle$ - arXiv:2209.10541

Using the W-states (or the kink W-states) we obtain a closed-form expression

$$\mathcal{M}_2(|W\rangle) = \mathcal{M}_2(|W_k\rangle) = 3\log_2(N) - \log_2(7N-6).$$

→ **first exact results for SRE to date**

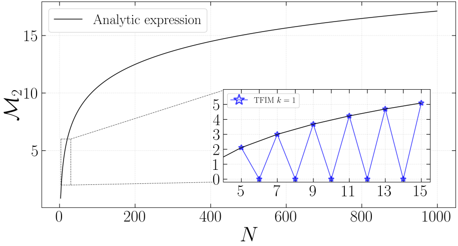


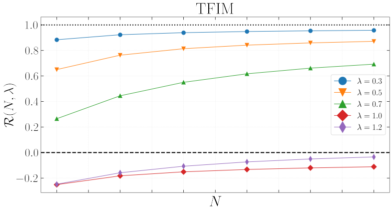
Figure: Analytics and numerics close to 'classical point' of the W-state

→ **valid beyond the classical point!**

4d. TD limit of TFIM magic - arXiv:2209.10541

- Universality: deep inside the quantum regime and in the thermodynamic limit!

$$\mathcal{R}(N,\lambda) = \frac{\mathcal{M}_2(1,N,\lambda) - \mathcal{M}_2(-1,N,\lambda)}{\mathcal{M}_2(|W\rangle)}.$$



The SRE of the frustrated phase has a non-local contribution

- Locality: SRE well approximated from local quantities such as local magnetization along the z-direction

$$\langle \sigma_j^z \rangle = m_z + \frac{2}{N}$$

$$\mathcal{M}_2(1,N,\lambda) \simeq N \log_2 \left(\frac{1+m_z^2}{1+m_z^4} \right) + 4m_z \left(\frac{1}{1+m_z^2} - \frac{2m_z^2}{1+m_z^4} \right)$$

- **Similar to the magic at QPT!**

The end!

Thank you for your attention!

Key points!

- effects of frustration on entanglement
- entanglement robustness
- quantum information perspective
- link between W and kink W state
- topological frustration induces magic!

Perspectives!

- Non-trivial topological properties of frustrated 1d chains
- finite momentum states magic