Spectral Form Factor and Random Walks

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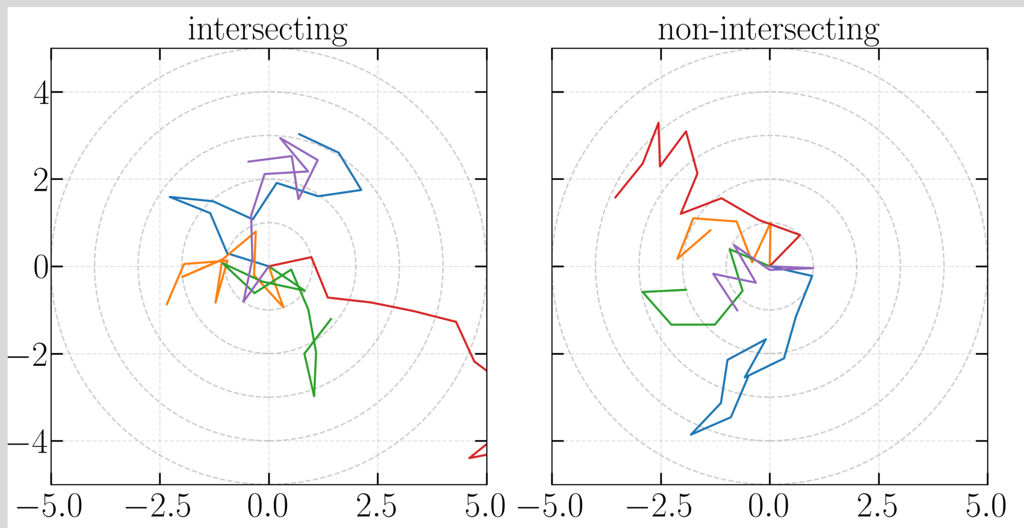
Abstract

This work examines the relationship between the *spectral form factor* (SFF) and *planar uniform random walks*. The SFF is a measure used to identify the onset of quantum chaos and scrambling in models that capture certain aspects of black holes [1]. We demonstrate that the moments of the SFF in generic quantum chaotic systems exhibit similar behavior to the moments of the distribution describing the positions of random walkers in the Euclidean plane [2]. This similarity holds for the mean, variance, skewness, kurtosis, and all higher-order moments of the underlying probability distribution function. Additionally, we suggest a potential generalization of planar random walks that avoid intersections along the trajectory and explore potential applications to integrable quantum systems.

Planar Uniform Random Walks

To illustrate this concept, consider a scenario where an intoxicated individual is stopped by law enforcement under suspicion of drunk driving. The police officer instructs the person to exit the vehicle and stand by the roadside. The officer then asks the individual to walk straight toward them for a common field sobriety test. However, due to severe intoxication, the person cannot maintain a straight path, and their steps become erratic and random.

A(I) drunk



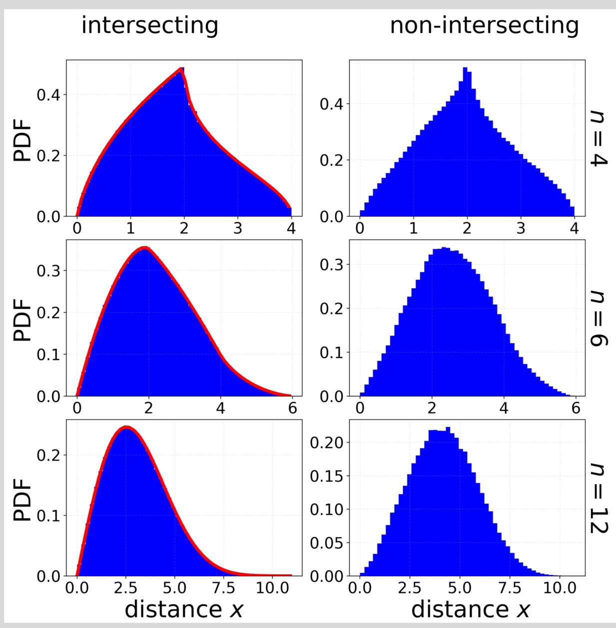
Computer simulations of 5 trajectories of length up 10

The mathematical problem of determining where a random walker - or a tipsy guy (hic!) - ends up after n steps relative to their starting point has been thoroughly explored. Long ago Kluyver [3] demonstrated that the *probability density function* (PDF) of the final distance in the intersecting case reads

$$\mathrm{PDF}(x,n) = \int_0^\infty J_0(xt)\,[J_0(t)]^n\, xt\;\mathrm{d}t \tag{1}$$

where J_0 is the Bessel function of the first kind and zeroth order defined $J_0(t) = (2/\pi)\int_0^{\pi/2}\cos(t\cos\varphi)\mathrm{d}\varphi$.

- The limiting $n \to \infty$ PDFs
  - intersecting case (known)
    $$\text{Rayleigh} = \frac{x}{\sigma^2}e^{-\frac{x^2}{2\sigma^2}}$$
  - non-intersecting case (not known?!)
    $$\text{Normal} = \frac{1}{\sqrt{2\pi\sigma^2}}e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$



10^6 trajectories

The *moments* of the distribution can be cast in a n-dimensional integral

$$W_n(s) := \int_0^n \mathrm{PDF}(t,n)t^s\;\mathrm{d}t \quad \text{or} \quad W_n(s) := \int_{[0,1]^n}\left|\sum_{k=1}^n e^{2\pi i x_k}\right|^s \mathrm{d}x, \tag{2}$$

and are known explicitly for the even moments and number of steps.

intersecting							non-intersecting					
n	$s=2$	$s=4$	$s=6$	$s=8$	$s=10$		n	$s=2$	$s=4$	$s=6$	$s=8$	$s=10$
2	2	6	20	70	252		2	2	6	20	71	256
3	3	15	93	639	4653		3	3	16	101	697	5077
4	4	28	256	2716	31504		4	5	34	313	3332	38701
5	5	45	545	7885	127905		5	6	60	755	11111	181544
6	6	66	996	18306	384156		6	8	97	1559	29593	-

Table: $W_n(s)$ at even integers. In red we denote new numerical estimates.

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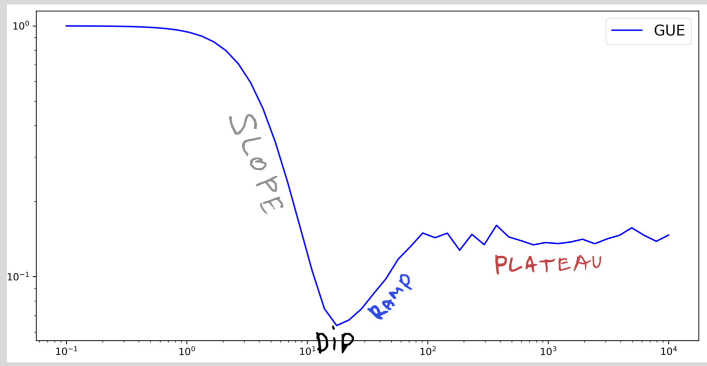
SCAN ME

Spectral Form Factor

The SFF is a valuable tool to probe discrete level spectrum in quantum chaotic systems [1]. It is defined via the partition function $Z(\beta) = \sum_n e^{-\beta E_n}$ as

$$\mathrm{SFF}(\beta,t) := \left|\frac{Z(\beta,t)}{Z(\beta)}\right|^2 = \frac{1}{Z(\beta)^2}\sum_{n,m} e^{-\beta(E_m+E_n)}e^{i(E_m-E_n)t}, \tag{3}$$

where $Z(\beta,t)$ is analytical continuation $\beta \to \beta + it$, and $\beta = 1/T$ is the inverse temperature.



SFF for GUE. $L=100, M=100$ and $\beta=5$.

- In quantum chaotic (non-integrable) systems whose spectrum is typically described by Random Matrix Theory (RMT) ensembles the SFF displays *slope-dip-ramp-plateau* behavior.
  - Sachdev-Ye-Kitaev (SYK) model displays such behavior [1].
- In non-chaotic (integrable) systems, the SFF has a *slope* and a *plateau*, but no linear ramp.
- The SFF is known to be *non-self averaging*, i.e. its probability distribution does not become a delta peak in the thermodynamic limit. $\longrightarrow$ ensemble average is important!

The connection

- We focus on the **infinite-temperature** version of the SFF, i.e. $T \to \infty$.
- A key assumption on the energy spectrum that we make is the so-called **no-resonance condition (NRC)** or non-degenerate energy gaps condition [4, 5]
  - Formally, under spectral decomposition of the Hamiltonian $H = \sum_{j=1}^d E_j|\varphi_j\rangle\langle\varphi_j|$. If H obeys NRC then
    $$E_m = E_n \tag{4}$$
    $$E_l + E_k = E_n + E_m \iff l=n, k=m; l=m, k=n \tag{5}$$
  - NRC condition is satisfied by quantum chaotic systems!
- We are interested in infinite-time average of the SFF moments defined as
  $$\overline{\mathrm{SFF}^p(\infty,t)} := \lim_{T\to\infty}\frac{1}{T}\int_0^T dt\mathrm{SFF}^p(\infty,t). \tag{6}$$
  - Taking the time average singles out the generalized NRC conditions under which
    $$E_{n_1} + E_{n_2}\cdots + E_{n_d} = E_{m_1} + E_{m_2}\cdots + E_{m_d} \tag{7}$$
  - Alternatively, a disorder average (subscript W) would do the same [6]
    $$\left\langle\left|\sum_{\alpha=1}^d e^{i2\pi E_\alpha t}\right|^{2p}\right\rangle_W = \left\langle\sum_{\alpha_1,\ldots,\alpha_p=1}^d\sum_{\beta_1,\ldots,\beta_p=1}^d e^{i2\pi\left(E_{\alpha_1}+\ldots+E_{\alpha_p}-E_{\beta_1}-\ldots-E_{\beta_p}\right)t}\right\rangle_W$$
    $$\approx \sum_{\alpha_1,\ldots,\alpha_p=1}^d\sum_{\beta_1,\ldots,\beta_p=1}^d\sum_{\Pi\in S_p}\delta_{(\alpha_1,\ldots,\alpha_p),(\beta_{\Pi_1},\ldots,\beta_{\Pi_p})} \tag{8}$$
- We numerically evaluate all the possible NRC conditions.
  - The number of conditions scales as a factorial p!
  - We obtain the same moments as for the intersecting planar uniform random walks!

Conclusion and Outlook

- We discovered a connection between the planar uniform random walks and the Spectral Form Factor (SFF) for the generic quantum chaotic systems!
- Is this framework extendable to non-generic and integrable models if additional constraints (such as non-intersection of trajectories) are assumed?

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